

# Coset enumeration for certain infinitely presented groups

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## Abstract

We describe an algorithm that computes the index of a finitely generated subgroup in a finitely  $L$ -presented group provided that this index is finite. This algorithm shows that the subgroup membership problem for finite index subgroups in a finitely  $L$ -presented group is decidable. As an application, we consider the low-index subgroups of some self-similar groups including the Grigorchuk group, the twisted twin of the Grigorchuk group, the Grigorchuk super-group and the Hanoi 3-group.

*Keywords:* Coset enumeration; recursive presentations; self-similar groups; Grigorchuk group; low-index subgroups.

## 1 Introduction

Many algorithmic problems are unsolvable for finitely presented groups in general. For instance, there is no algorithm which allows to decide if a group given by a finite presentation is trivial [21]. However, the coset enumeration process introduced by Todd & Coxeter [27] and investigated by various others, see [19] or the historical notes in Chapter 5.9 of [26], computes the index of a finitely generated subgroup in a finitely presented group provided that this index is finite. Therefore, the Todd-Coxeter method allows one to prove that a finitely presented group is trivial.

Coset enumeration is one of the most important tools for investigating finitely presented groups; but, if the subgroup has infinite index, this process will not terminate. Even if the subgroup has finite index, there is no upper bound on the complexity of coset enumeration. Therefore, even proving a finitely presented group being trivial is computationally a challenging problem [19, 23].

For this reason, solving algorithmic problems for infinitely presented groups seems entirely infeasible. However, an interesting family of recursively presented groups was recently shown to be applicable for computer investigations. Examples of such groups arise as subgroups of the automorphism group of a regular tree. A famous example is the Grigorchuk group  $\mathfrak{G}$  which plays a prominent role in the area of Burnside problems [16]. The group  $\mathfrak{G}$  is finitely generated and it admits a recursive presentation whose relations are given recursively

by the action of a finitely generated free monoid of endomorphisms acting on finitely many relations [22]. Infinite presentations of this type are called *finite L-presentations* in honor of Lysénok’s latter result for the Grigorchuk group  $\mathfrak{G}$ ; see Section 2 or [1] for a definition.

Finite  $L$ -presentations are ‘natural’ generalizations of finite presentations and, as the concept is quite general, they found their application in various aspects of group theory; see, for instance, [9, 20, 24]. A finite  $L$ -presentation of a group allows to compute its lower central series quotients [3] and the Dwyer quotients of its Schur multiplier [18]. The Dwyer quotients often exhibit periodicities which yield detailed information on the structure of the Schur multiplier in general.

In this paper, we describe a coset enumeration process for computing the index of a finitely generated subgroup in a finitely  $L$ -presented group provided that this index is finite. In order to achieve this method, we show in Section 3 that finitely many relations are sufficient to compute an upper bound on the index using coset enumeration for finitely presented groups. It then remains to either prove that this upper bound is sharp or to improve the bound otherwise. In Section 4, we show that the latter problem is algorithmically decidable in general. In particular, we show that there exists an algorithm which decides whether or not a map from the free group over the  $L$ -presentations generators into a finite group induces a homomorphism from the  $L$ -presented group.

Similar to coset enumeration for finitely presented groups, our method for finitely  $L$ -presented groups allows straightforward applications including a membership test for finite index subgroups. In particular, our method allows us to compute the number of subgroups with small index for some self-similar groups in Section 5. Our explicit computations correct the counts obtained in [5, 4], and hence we provide a further step towards Problem 6.1 raised in [14].

We have implemented our coset enumeration method and its applications in the computer algebra system GAP [13]. Computer experiments with this implementation demonstrate that our method works reasonably well in practice.

In a forthcoming paper, we prove a variant of the Reidemeister-Schreier theorem for finitely  $L$ -presented groups which shows that each finite index subgroup of a finitely  $L$ -presented group is finitely  $L$ -presented itself.

## 2 Preliminaries

We briefly recall the notion of a finite  $L$ -presentation as introduced in [1]. For this purpose, let  $F$  be a finitely generated free group over the alphabet  $\mathcal{X}$ . Furthermore, let  $\mathcal{Q}, \mathcal{R} \subset F$  and  $\Phi \subset \text{End}(F)$  be finite subsets. Then the quadruple  $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$  is a *finite L-presentation*. It defines the *finitely L-presented group*

$$G = \left\langle \mathcal{X} \mid \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \right\rangle, \quad (1)$$

where  $\Phi^*$  denotes the free monoid of endomorphisms generated by  $\Phi$ ; that is, the closure of  $\{\text{id}\} \cup \Phi$  under composition of endomorphisms. We will also write

$G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$  for the finitely  $L$ -presented group in Eq. (1).

Clearly, every finitely presented group  $\langle \mathcal{X} \mid \mathcal{R} \rangle$  is finitely  $L$ -presented by  $\langle \mathcal{X} \mid \emptyset \mid \emptyset \mid \mathcal{R} \rangle$ . Therefore, finite  $L$ -presentations generalize the concept of finite presentations. Other examples of finitely  $L$ -presented groups are various self-similar groups or branch groups [1]. For instance, the Grigorchuk group satisfies the following

**Theorem 1 (Lysënok, 1985)** *The Grigorchuk group  $\mathfrak{G}$  is finitely  $L$ -presented by*

$$\langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bcd\} \mid \{\sigma\} \mid \{(ad)^4, (adacac)^4\} \rangle,$$

where  $\sigma$  is the endomorphism of the free group over the alphabet  $\{a, b, c, d\}$  induced by the mapping  $a \mapsto aca$ ,  $b \mapsto d$ ,  $c \mapsto b$ , and  $d \mapsto c$ .

*Proof.* For a proof, we refer to [22]. □

Finite  $L$ -presentations are recursive presentations which are ‘natural’ generalizations of finite presentations. They were used by various authors to construct groups with interesting properties; see, for instance, [9, 20, 24]. Furthermore, every free group in a variety of groups that satisfies finitely many identities is finitely  $L$ -presented [1]; e.g., the free Burnside group  $B(n, m)$  of exponent  $m$  on  $n$  generators is finitely  $L$ -presented by

$$\langle \{a_1, \dots, a_n\} \cup \{t\} \mid \{t\} \mid \Sigma \mid \{t^m\} \rangle,$$

where the endomorphisms  $\Sigma = \{\sigma_x \mid x \in \{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}\}$  are induced by the mappings

$$\sigma_x: \begin{cases} a_i & \mapsto a_i, & \text{for each } 1 \leq i \leq n \\ t & \mapsto tx, \end{cases}$$

for each  $x \in \{a_1^{\pm 1}, \dots, a_n^{\pm 1}\}$ .

### 3 Coset enumeration for finitely $L$ -presented groups

Let  $G = \langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$  be a finitely  $L$ -presented group and let  $\mathcal{U} \leq G$  be a finitely generated subgroup with finite index in  $G$ . In this section, we show that coset enumeration for finitely presented groups yields an upper bound on the index  $[G : \mathcal{U}]$ . In Section 4, it then remains to prove (or disprove) that this upper bound is sharp.

Let  $\{g_1, \dots, g_n\}$  be a generating set for the subgroup  $\mathcal{U}$ . We assume that the generators of  $\mathcal{U}$  are given as words over the alphabet  $\mathcal{X} \cup \mathcal{X}^-$ . Denote the free group over  $\mathcal{X}$  by  $F$  and let  $K$  be the normal subgroup

$$K = \left\langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^*} \mathcal{R}^\sigma \right\rangle^F.$$

so that  $G \cong F/K$  holds. Then the subgroup  $E = \langle g_1, \dots, g_n \rangle \leq F$  satisfies that  $\mathcal{U} \cong EK/K$ . Hence, we are to compute the index  $[G : \mathcal{U}] = [F : EK]$ .

For an element  $\sigma \in \Phi^*$ , we denote by  $\|\sigma\|$  the usual word-length in the generating set  $\Phi$  of the free monoid  $\Phi^*$ . Define  $\Phi^{(i)} = \{\sigma \in \Phi^* \mid \|\sigma\| \leq i\}$ , for each  $i \in \mathbb{N}_0$ . Then, as  $\mathcal{Q}$ ,  $\Phi$ , and  $\mathcal{R}$  are finite sets, the normal subgroup

$$K_i = \left\langle \mathcal{Q} \cup \bigcup_{\sigma \in \Phi^{(i)}} \mathcal{R}^\sigma \right\rangle^F$$

is finitely generated as normal subgroup. We obtain  $K = \bigcup_{i \geq 0} K_i$  and also  $EK = \bigcup_{i \geq 0} EK_i$ . Consider the ascending chain of subgroups

$$EK_0 \leq EK_1 \leq \dots \leq EK_\ell \leq EK_{\ell+1} \leq \dots \leq EK \leq F.$$

Then the following lemma is straightforward.

**Lemma 2** *The subgroup  $EK$  has finite index in  $F$  if and only if there exists  $\ell \in \mathbb{N}$  such that  $EK_\ell$  has finite index in  $F$ . In that case, there exists  $\ell' \in \mathbb{N}$  such that  $EK_{\ell'} = EK$ .*

*Proof.* Obviously, if  $[F : EK_\ell]$  is finite for some  $\ell \in \mathbb{N}$ , then the subgroup  $EK$  has finite index in  $F$ . On the other hand, if  $[F : EK]$  is finite, then, as  $F$  is finitely generated, the subgroup  $EK$  is finitely generated. Let  $\{u_1, \dots, u_n\}$  be a generating set of  $EK$ . Since  $EK = \bigcup_{i \geq 0} EK_i$  holds, there exists a positive integer  $\ell \in \mathbb{N}$  such that  $\{u_1, \dots, u_n\} \subseteq EK_\ell$  and thus  $EK_\ell = EK$ .  $\square$

Note that the index  $[F : EK_\ell]$  is the index of the subgroup  $\mathcal{U}$  in the finitely presented *covering group*

$$G_\ell = \langle \mathcal{X} \mid \{q, r^\sigma \mid q \in \mathcal{Q}, r \in \mathcal{R}, \sigma \in \Phi^{(\ell)}\} \rangle. \quad (2)$$

By Lemma 2, there exists a positive integer  $\ell \in \mathbb{N}$  so that the subgroup  $\mathcal{U}$  has finite index in  $G_\ell$ . In this case, coset enumeration for finitely presented groups computes the index  $[G_\ell : \mathcal{U}]$ . Although we do not know this integer  $\ell$  *a priori*, we can use the following firsthand approach to find such an integer: Starting with  $\ell = 1$ , we attempt to prove finiteness of  $[G_\ell : \mathcal{U}]$  using coset enumeration for finitely presented groups. If this attempt does not succeed within a previously defined time limit, we increase the integer  $\ell$  and the time limit. We continue this process until eventually the index  $[G_\ell : \mathcal{U}]$  is proved to be finite. In theory, Lemma 2 guarantees that this process will terminate. Computer experiments with the implementation of our method in GAP show that this firsthand approach works reasonably well in practice. In particular, our implementation allows to compute the index of all subgroups considered in [2, 4, 14] and Chapter VIII of [10].

Suppose that the integer  $\ell \in \mathbb{N}$  is chosen so that  $n = [G_\ell : \mathcal{U}]$  is finite and that the coset enumeration for finitely presented groups has terminated and has computed a permutation representation  $\varphi_\ell: F \rightarrow \mathcal{S}_n$  for the group's action on the right cosets  $EK_\ell \backslash F$ . Then the index  $[G : \mathcal{U}] = [F : EK]$  divides the index  $[G_\ell : \mathcal{U}] = [F : EK_\ell]$ , and hence  $[G_\ell : \mathcal{U}]$  is an upper bound on  $[G : \mathcal{U}]$ . It therefore remains to either prove that  $[F : EK] = [F : EK_\ell]$  holds, or to

increase the integer  $\ell$  otherwise. The permutation representation  $\varphi_\ell: F \rightarrow \mathcal{S}_n$  is called *valid*, if  $[F : EK] = [F : EK_\ell]$  holds.

Clearly, a permutation representation  $\varphi_\ell: F \rightarrow \mathcal{S}_n$  is valid if and only if every relation  $r \in F$  of the group presentation is contained in the kernel of  $\varphi_\ell$ . Therefore, if the group  $G = F/K$  were finitely presented, only finitely many relations need to be considered to prove validity of  $\varphi_\ell$ . However, for finitely  $L$ -presented groups, even checking validity of a permutation representation  $\varphi_\ell$  involves possibly infinitely many relations. In Section 4, we prove that the latter problem is decidable in general.

## 4 Deciding validity of a permutation representation

In this section, we describe our algorithm for deciding whether or not a permutation representation  $\varphi: F \rightarrow \mathcal{S}_n$ , as considered in Section 3, is valid. This is equivalent to checking whether a coset-table for  $\mathcal{U}$  in  $G_\ell$  obtained by the methods of Section 3 defines the given subgroup  $\mathcal{U} \leq G$ .

Let  $\varphi: F \rightarrow \mathcal{S}_n$  be a permutation representation as in Section 3 and let  $\Phi^*$  be the free monoid generated by a finite set  $\Phi \subseteq \text{End}(F)$ . For two endomorphisms  $\sigma \in \Phi^*$  and  $\delta \in \Phi^*$ , we say that  $\delta$  *reduces to  $\sigma$  with respect to  $\varphi$*  if there exists a homomorphism  $\pi: \text{im}(\sigma\varphi) \rightarrow \text{im}(\delta\varphi)$  such that  $\sigma\varphi\pi = \delta\varphi$ . In this case, we will write  $\delta \rightsquigarrow_\varphi \sigma$ . Note that  $\rightsquigarrow$  is a reflexive and transitive relation on the endomorphisms  $\Phi^*$ . The following lemma gives an equivalent definition for  $\delta \rightsquigarrow_\varphi \sigma$ .

**Lemma 3** *Let  $\delta, \sigma \in \text{End}(F)$  be given. Then  $\delta$  reduces to  $\sigma$  with respect to  $\varphi$  if and only if  $\ker(\sigma\varphi) \leq \ker(\delta\varphi)$  holds.*

*Proof.* Assume that  $\delta \rightsquigarrow_\varphi \sigma$  holds. Then, by definition, there exists a homomorphism  $\pi: \text{im}(\sigma\varphi) \rightarrow \text{im}(\delta\varphi)$  such that  $\sigma\varphi\pi = \delta\varphi$ . Let  $g \in \ker(\sigma\varphi)$ . Then we have that  $g^{\delta\varphi} = g^{\sigma\varphi\pi} = (g^{\sigma\varphi})^\pi = 1$  and hence, we obtain  $g \in \ker(\delta\varphi)$ . Suppose that  $\ker(\sigma\varphi) \leq \ker(\delta\varphi)$  holds. Then we have the isomorphisms  $F/\ker(\sigma\varphi) \rightarrow \text{im}(\sigma\varphi)$ ,  $g \ker(\sigma\varphi) \mapsto g^{\sigma\varphi}$  and  $F/\ker(\delta\varphi) \rightarrow \text{im}(\delta\varphi)$ ,  $g \ker(\delta\varphi) \mapsto g^{\delta\varphi}$ . We further have the natural homomorphism  $F/\ker(\sigma\varphi) \rightarrow F/\ker(\delta\varphi)$ ,  $g \ker(\sigma\varphi) \mapsto g \ker(\delta\varphi)$ . This yields the existence of a homomorphism  $\pi: \text{im}(\sigma\varphi) \rightarrow \text{im}(\delta\varphi)$  such that  $g^{\sigma\varphi\pi} = g^{\delta\varphi}$ .  $\square$

A finite generating set for the kernel  $\ker(\sigma\varphi)$  is given by the Schreier theorem [21, Proposition 3.7] and hence, it is straightforward to check whether or not  $\delta \rightsquigarrow_\varphi \sigma$  holds. The definition  $\delta \rightsquigarrow_\varphi \sigma$  also yields the following immediate consequence.

**Lemma 4** *There is no infinite set of endomorphisms of  $F$  such that for each pair  $(\sigma, \delta)$  from this set, neither  $\sigma \rightsquigarrow_\varphi \delta$  nor  $\delta \rightsquigarrow_\varphi \sigma$  hold.*

*Proof.* Obviously, for every endomorphism  $\sigma \in \text{End}(F)$ , it holds that  $\sigma \rightsquigarrow_\varphi \sigma$ . By the universal property of the free group  $F$ , a homomorphism  $\sigma\varphi: F \rightarrow \mathcal{S}_n$  is uniquely defined by the images  $x_1^{\sigma\varphi}, \dots, x_n^{\sigma\varphi}$  of the elements  $x_1, \dots, x_n$  of a basis of  $F$ . Since  $\text{im}(\varphi)$  is a finite group, there are only finitely many homomorphisms

$F \rightarrow \text{im}(\varphi)$  and therefore  $\text{Hom}(F, \text{im}(\varphi))$  is finite. Hence, an infinite set of endomorphisms will contain endomorphisms  $\sigma$  and  $\delta$  with  $x_i^{\sigma\varphi} = x_i^{\delta\varphi}$ , for each  $1 \leq i \leq n$ . In this case,  $\sigma \rightsquigarrow_\varphi \delta$  obviously holds.  $\square$

An element  $\sigma \in \Phi^*$  is called a  $\Phi$ -descendant of  $\delta \in \Phi^*$  if there exists  $\psi \in \Phi$  such that  $\sigma = \psi\delta$ . Thereby, the free monoid  $\Phi^*$  obtains the structure of a  $|\Phi|$ -regular rooted tree with its root being the identity map  $\text{id}: F \rightarrow F$ . We can further endow the monoid  $\Phi^*$  with a length-plus-(from the right)-lexicographic ordering  $\prec$  by choosing an arbitrary ordering on the finite set  $\Phi$ . More precisely, we define  $\sigma \prec \delta$  if  $\|\sigma\| < \|\delta\|$  or, otherwise, if  $\sigma = \sigma_1 \cdots \sigma_n$  and  $\delta = \delta_1 \cdots \delta_n$ , with  $\sigma_i, \delta_i \in \Phi$ , and there exists a positive integer  $1 \leq k \leq n$  such that  $\sigma_i = \delta_i$ , for  $k < i \leq n$ , and  $\sigma_k \prec \delta_k$ . Since  $\Phi$  is finite, the obtained ordering  $\prec$  is a well-ordering on the monoid  $\Phi^*$ , see [26], and therefore there is no infinite  $\prec$ -descending series of endomorphisms in  $\Phi^*$ .

Our algorithm that decides validity of a permutation representation  $\varphi: F \rightarrow \mathcal{S}_n$  is displayed in Algorithm 1 below. We need to prove the following

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ISVALIDPERMREP( $\mathcal{X}, \mathcal{Q}, \Phi, \mathcal{R}, \mathcal{U}, \varphi$ )
  Initialize  $V := \{\text{id}: F \rightarrow F\}$  and  $S := \Phi$ .
  Choose an ordering on  $\Phi = \{\phi_1, \dots, \phi_n\}$  with  $\phi_i \prec \phi_{i+1}$ .
  while  $S \neq \emptyset$  do
    Remove the first entry  $\delta$  from  $S$ .
    if  $(\exists r \in \mathcal{R}: r^\delta \notin \ker \varphi)$  then return( false )
    if not  $(\exists \sigma \in V: \delta \rightsquigarrow_\varphi \sigma)$  then
      Append  $\phi_1\delta, \dots, \phi_n\delta$  to  $S$ .
      Add  $\delta$  to  $V$ .
  return( true )

```

Algorithm 1: Deciding validity of a permutation representation

**Theorem 5** *The algorithm ISVALIDPERMREP returns **true** if and only if the permutation representation  $\varphi: F \rightarrow \mathcal{S}_n$  is valid.*

*Proof.* The ordering  $\prec$  on  $\Phi$  can be extended to an ordering on  $\Phi^*$  as described above. By construction, the stack  $S$  is ordered with respect to  $\prec$ . Since  $F$  is finitely generated, the set of homomorphisms  $\text{Hom}(F, \mathcal{S}_n)$  is finite. Thus, in particular, the set  $\{\delta\varphi \mid \delta \in V\} \subseteq \text{Hom}(F, \mathcal{S}_n)$  is finite and therefore the algorithm ISVALIDPERMREP can add only finitely many endomorphisms to the set  $V$ . Thus, for every  $\Phi$ -descendant  $\delta$  in the stack  $S$ , there will eventually exist an element  $\sigma \in V$  such that  $\delta \rightsquigarrow_\varphi \sigma$ . Therefore, the algorithm ISVALIDPERMREP is guaranteed to terminate and it returns either **true** or **false**. Clearly, if the algorithm returned **false**, then it found a relation  $r^\delta$  which yields a coincidence, and hence the permutation representation  $\varphi: F \rightarrow \mathcal{S}_n$  is not valid.

Suppose that the algorithm returned **true**. By the constructions of Section 3, the fixed relations in  $\mathcal{Q}$  of the  $L$ -presentation  $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$  are already contained in the kernel of the permutation representation  $\varphi$ . Therefore, it suffices to prove that every relation of the form  $r^{\sigma^1}$ , with  $r \in \mathcal{R}$  and

$\sigma_1 \in \Phi^*$ , is contained in the kernel of  $\varphi$ . By construction, there exists  $\delta \in V$  maximal subject to the existence of  $w \in \Phi^*$  such that  $\sigma_1 = w\delta$ . If  $\|w\| = 0$ , then  $\sigma_1 = \delta$  is contained in  $V$  and therefore  $r^\delta \in \ker \varphi$ , as the algorithm did not return **false**. Otherwise, there exist  $\psi \in \Phi$  and  $v \in \Phi^*$  such that  $w\delta = v\psi\delta$ . Since  $\psi\delta \notin V$ , there exists an element  $\varepsilon \in V$  with  $\varepsilon \prec \psi\delta$ , by construction, such that  $\psi\delta$  reduces to  $\varepsilon$  with respect to  $\varphi$ . Thus, by definition, there exists a homomorphism  $\pi: \text{im}(\varepsilon\varphi) \rightarrow \text{im}(\psi\delta\varphi)$  such that  $\psi\delta\varphi = \varepsilon\varphi\pi$ . In particular, we obtain that  $r^{\sigma_1\varphi} = r^{w\delta\varphi} = r^{v\psi\delta\varphi} = r^{v\varepsilon\varphi\pi}$ . As  $\pi$  is a homomorphism, it suffices to prove that  $r^{v\varepsilon} \in \ker \varphi$ . Note that, since  $\varepsilon \prec \psi\delta$ , we have that  $v\varepsilon \prec v\psi\delta = \sigma_1$ . Continuing this rewriting process with the element  $\sigma_2 = v\varepsilon$  yields a descending sequence  $\sigma_1 \succ \sigma_2 \succ \dots$  in the monoid  $\Phi^*$ . As the ordering  $\prec$  is a well-ordering, this process terminates with an element  $\sigma_n \in V$ . Since the algorithm did not return **false**, we have that  $r^{\sigma_n} \in \ker \varphi$  which proves the assertion.  $\square$

Note that, if the algorithm **ISVALIDPERMREP** found a coincidence, this can be used to update the coset-table, and thus another application of coset enumeration for finitely presented groups can be avoided. Moreover, the Algorithm 1 yields the following

**Theorem 6** *Let  $G$  be finitely  $L$ -presented by  $\langle \mathcal{X} \mid \mathcal{Q} \mid \Phi \mid \mathcal{R} \rangle$  and denote the free group over  $\mathcal{X}$  by  $F$ . There exists an algorithm which decides whether or not a homomorphism  $\varphi: F \rightarrow \mathcal{S}_n$  induces a homomorphism  $G \rightarrow \mathcal{S}_n$ .*

If  $\Phi^* = \{\sigma\}^*$  is generated by a single element  $\sigma \in \text{End}(F)$ , then there will exist positive integers  $0 \leq i < j$  such that  $\sigma^j \sim_\varphi \sigma^i$ . In this case, the algorithm **ISVALIDPERMREP** simplifies to the following

**Corollary 7** *Let  $0 \leq i < j$  be positive integers such that  $\sigma^j \sim_\varphi \sigma^i$ . Then we have  $[F : EK_\ell] = [F : EK]$  if and only if*

$$\{q, r^{\sigma^k} \mid q \in \mathcal{Q}, r \in \mathcal{R}, 0 \leq k < j\} \subseteq \ker \varphi. \quad (3)$$

We consider the following

**Example 8** *Let  $G$  denote the Basilica Group [17]. Then  $G$  is finitely  $L$ -presented by  $\langle \{a, b\} \mid \emptyset \mid \{\sigma\} \mid \{[a, a^b]\} \rangle$ , where  $\sigma$  is induced by the mapping  $a \mapsto b^2$  and  $b \mapsto a$ ; see [8]. We consider the subgroup  $\mathcal{U} = \langle a^3, b, aba \rangle$ . A coset enumeration for finitely presented groups yields that the subgroup  $\mathcal{U}$  has index 3 in the finitely presented covering group*

$$G_0 = \langle \{a, b\} \mid \{[a, a^b]\} \rangle.$$

Furthermore, we obtain the permutation representation  $\varphi: F \rightarrow \mathcal{S}_3$  for the group's action on the cosets  $EK_0 \setminus F$ . This permutation representation is induced by the mapping

$$a \mapsto (1, 2, 3) \quad \text{and} \quad b \mapsto (2, 3).$$

We now obtain the images

$$\begin{aligned} a^{\sigma\varphi} &= (), & b^{\sigma\varphi} &= (1, 2, 3), \\ a^{\sigma^2\varphi} &= (1, 3, 2), & b^{\sigma^2\varphi} &= (), \\ a^{\sigma^3\varphi} &= (), & b^{\sigma^3\varphi} &= (1, 3, 2). \end{aligned}$$



Clearly, the mapping  $a^{\sigma\varphi} \mapsto a^{\sigma^3\varphi}$  and  $b^{\sigma\varphi} \mapsto b^{\sigma^3\varphi}$  induces a homomorphism  $\pi: \text{im}(\sigma\varphi) \rightarrow \text{im}(\sigma^3\varphi)$ , and hence we have  $\sigma^3 \sim_{\varphi} \sigma$ . By Corollary 7, it therefore suffices to prove that

$$([a, a^b])^{\varphi} = ( ), \quad ([a, a^b])^{\sigma\varphi} = ( ), \quad \text{and} \quad ([a, a^b])^{\sigma^2\varphi} = ( )$$

hold. This yields that  $[G : \mathcal{U}] = 3$ .

## 5 Further applications

The permutation representation  $\varphi: F \rightarrow \mathcal{S}_n$  for a finite index subgroup  $EK/K \leq F/K$  yields various algorithmic applications. For instance, an element  $w \in F$  is contained in the given subgroup  $EK$  if and only if it stabilizes the trivial coset  $EK1$ . This can be easily be checked using the permutation representation  $\varphi$ . In particular, we obtain

**Theorem 9** *The subgroup membership problem for finite index subgroups in a finitely  $L$ -presented group is decidable.*

Moreover, having computed permutation representations  $\varphi_1$  and  $\varphi_2$  for two finite index subgroups  $U$  and  $V$  of a finitely  $L$ -presented group, one can compute a generating set for the intersection  $U \cap V$ . Thus, in particular, our method allows one to compute the core of a finite index subgroup. For example, the core of the subgroup  $\mathcal{U}$  in Example 8 is given by

$$H = \langle b^2, a^3, a^2ba^{-1}b^{-1}, abab^{-1}, ab^2a^{-1}, ba^2b^{-1}a^{-1}, baba^{-2} \rangle.$$

Since  $H$  has finite index in  $G$ , our method allows to compute a permutation representation for the core  $H$  and we obtain  $G/H \cong \mathcal{S}_3$ .

### 5.1 Low-index subgroups of finitely $L$ -presented groups

The coset enumeration process for finitely presented groups was used in [11] to describe a low-index subgroup algorithm that computes all subgroups of a finitely presented group up to a given index. This algorithm also yields a method for computing all subgroups with small index in a finitely  $L$ -presented group. In this section, we will describe this method for finitely  $L$ -presented groups and we use this algorithm to investigate some self-similar groups. In particular, our implementation in the computer algebra system GAP allows us to determine the number of subgroups with index at most 64 in the Grigorchuk group.

Let  $G = F/K$  be a finitely  $L$ -presented group and let  $n$  be a positive integer. Using the low-index subgroup algorithm for finitely presented groups [11], see also Chapter 5.6 of [26], we obtain the list of subgroups with index at most  $n$  in the finitely presented covering group  $G_\ell = F/K_\ell$ . Since the covering group  $G_\ell$  naturally maps onto  $G$ , every subgroup  $EK_\ell/K_\ell$  with index  $n$  in  $G_\ell$  maps to a subgroup of the finitely  $L$ -presented group  $G$ . The index of this image  $EK/K$  in  $G$  divides the index  $n = [F : EK_\ell]$ . On the other hand, every subgroup  $EK/K$



with index  $n$  in the finitely  $L$ -presented group  $F/K$  has a full preimage  $EK/K_\ell$  in the finitely presented group  $G_\ell$  with index  $n$ . Thus the list of subgroups with index at most  $n$  in a finitely  $L$ -presented group  $G$  can be obtained from the list of subgroups of a finitely presented covering group  $G_\ell$  by removing duplicate images. Our solution to the subgroup membership problem can be used to remove duplicate images in  $G$ .

As an application, we consider some interesting self-similar groups and we determine the number of subgroups with small index. We first consider the Grigorchuk group  $\mathfrak{G}$ : its lattice of normal subgroups is well-understood [2] while its lattice of subgroups with finite index is widely unknown [14]. It is well known [14] that the Grigorchuk group has seven subgroups of index two. In [25], it was shown that the subgroups of index two are the only maximal subgroups of  $\mathfrak{G}$ . Our low-index subgroup algorithm allows us to determine the number of subgroups with index at most 64 in the group  $\mathfrak{G}$  and thereby, it corrects the counts in Section 7.4 of [5] and in Section 4.1 of [4]. The following list summarizes the number of subgroups ( $\leq$ ) and the number of normal subgroups ( $\trianglelefteq$ ) among them:

index	1	2	4	8	16	32	64
$\leq$	1	7	31	183	1827	22931	378403
$\trianglelefteq$	1	7	7	7	5	3	3

The Grigorchuk super-group  $\tilde{\mathfrak{G}}$  was introduced in [4]. It contains the Grigorchuk group as an infinite index subgroup. Little is known about its subgroup lattice. The twisted twin  $\bar{\mathfrak{G}}$  of the Grigorchuk group was introduced in [6]. Similarly, little is known about the subgroup lattice of the twisted twin  $\bar{\mathfrak{G}}$ . Our low-index subgroup algorithm allows us to determine the number of subgroups with index at most 16 in both groups. Their subgroup counts are:

index	$\tilde{\mathfrak{G}}$		$\bar{\mathfrak{G}}$	
	$\leq$	$\trianglelefteq$	$\leq$	$\trianglelefteq$
1	1	1	1	1
2	15	15	15	15
4	147	35	147	35
8	2163	43	1963	43
16	52403	55	46723	47

As both groups are 2-groups, the only maximal subgroups with finite index are the subgroups with index two; though the question of determining all maximal subgroups of  $\tilde{\mathfrak{G}}$  and  $\bar{\mathfrak{G}}$  has not been addressed in this paper.

Finally, we consider the Basilica group and the Hanoi-3 group [15] with its  $L$ -presentation from [7]. The following list also includes the number of maximal

subgroups (max):

index	Hanoi-3			Basilica		
	$\leq$	$\trianglelefteq$	max	$\leq$	$\trianglelefteq$	max
1	1	1	1	1	1	1
2	7	7	7	3	3	3
3	12	0	12	7	4	7
4	59	7	4	19	7	0
5	15	0	15	11	6	11
6	136	4	0	39	13	0
7	21	0	21	15	8	15
8	335	13	0	163	19	0
9	225	0	0	115	13	9
10	153	3	0	83	19	0
11	33	0	33	23	12	23
12	2872	12	0	355	31	0
13	39	0	39	27	14	27
14	297	3	0	115	25	0
15	450	0	0	77	24	0
16	1855	13	0	1843	47	0
17	51	0	51	35	18	35
18	5001	3	0	1047	44	0
19	57	0	57	39	20	39
20	1189	9	0	939	45	0
21	756	0	0	105	32	0
22	531	3	0	223	37	0
23	69	0	69	47	24	47
24	52220	23	0	4723	87	0
25	225	0	75	411	31	25
26	783	3	0	315	43	0
27	5616	0	27	736	49	0
28	2301	9	0			
29	87	0	87			
30	15462	3	0			
31	93	0	93			
32	9119	25	0			

The largest abelian quotient  $H/H'$  of the Hanoi-3 group  $H$  is 2-elementary abelian of rank 3. Thus, by the Feit-Thompson theorem [12], there are no normal subgroups with odd index in the Hanoi-3 group.

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